Fractional statistics of topological defects in graphene and related structures

B. Seradjeh[1] and M. Franz[1,2]

[1]Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada V6T 1Z1
[2]Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106

(Dated: February 2, 2008)

We show that fractional charges bound to topological defects in the recently proposed time-reversal-invariant models on honeycomb and square lattices obey fractional statistics. The effective low-energy description is given in terms of a 'doubled' level-2 Chern-Simons field theory, which is parity and time-reversal invariant and implies two species of semions (particles with statistical angle $\pm\pi/2$) labeled by a new emergent quantum number that we identify as the fermion axial charge.

Introduction – When the excitations of a many-body system carry electric charge that is smaller than the charge of its constituent particles (e.g., electrons) the charge is said to be *fractionalized*. This phenomenon is known to occur in the fractional quantum Hall (FQH) liquids [1], quintessential strongly correlated systems with broken time-reversal symmetry, where fractionally-charged excitations also obey fractional exchange statistics [2]. Recently, two model systems have been introduced on the honeycomb and square lattices [3, 4] that exhibit charge fractionalization *without breaking of the time-reversal symmetry*. These models, in essence, generalize the concept of fractionalization in polyacetylene [5] to two dimensions and, remarkably, can be considered weakly correlated. The experience with FQH systems suggests that the exchange statistics of these fractionally charged excitations could be anomalous. This question is interesting for several reasons. First, a very general argument can be made [6] that would seem to prohibit the existence of anyons in systems that obey time-reversal symmetry. Second, fractional statistics have recently captured attention due to their relevance to topologically protected quantum information processing [7]. Since the honeycomb lattice is found in natural graphene [8], and the square lattice model could be realized in artificially engineered structures [9], the possibility of realizing anyons in time-reversal invariant systems has both theoretical and practical significance.

In this Letter we construct the low-energy effective theory for the fractional particles in models [3, 4]. We find that they are indeed anyons, albeit of a very special kind, described by a *doubled* $\mathrm{U}(1)_2 \times \overline{\mathrm{U}(1)}_2$ Chern-Simons (CS) theory previously discussed by Freedman *et al.* [10]. In its topological sector the theory contains two species of semions, which transform into each other under parity and time reversal, thus escaping the constraints imposed by the argument of Ref. 6. Systems under consideration here [3, 4] represent the first explicit example of models for which such a gauge structure emerges as the low-energy effective theory.

Fractional charge – The low-energy theory for fermions on the graphene honeycomb lattice [11] and the square lattice threaded with π flux per plaquette [12] is the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ + me^{-i\chi\gamma_5}\right)\psi \tag{1}$$

where $z\!\!\!/ = \gamma_\mu z_\mu$, $\mu = 0, 1, 2$, γ_μ are 4×4 Dirac matrices in the Weyl representation and ψ is a four-component Dirac spinor whose components index two Dirac points ('valleys') and two sublattices. The mass m arises from the dimerization of hopping amplitudes introduced in Refs. 3, 4. The phase χ describes the direction of the dimerization pattern. Hou *et al.* [3] made a remarkable discovery that a topological defect (a vortex in χ) binds fractional charge $\pm e/2$. Determination of this fractional charge relies on establishing the existence of unpaired zero modes in the solution of the associated Dirac equation [13, 14, 15]. For our discussion it is useful to deduce the fractional charge by a method that does not rely on zero modes but instead exploits the long-distance behavior of Dirac fermions in topologically non-trivial backgrounds.

The idea here, originally due to Goldstone and Wilczek [16], is to follow the flow of charge as we introduce a vortex into χ by adiabatically deforming the order parameter, starting from a uniform configuration, e.g., $\chi(x) = 0$. This adiabatic insertion is made possible by temporarily enlarging the parameter space of masses by adding to the Lagrangian (1) a "γ_3-mass" term, $m_3\bar{\psi}\gamma_3\psi$. Physically this term corresponds to staggered on-site potential $\pm m_3$ for the fermions on two sublattices of the square or honeycomb lattice. Such a term could in principle appear in the physical Lagrangian but we add it here by hand to enlarge the symmetry of the order parameter from U(1) to O(3). Let us denote this O(3) order parameter by the vector

$$\boldsymbol{\varphi}(x) = (m\cos\chi, m\sin\chi, m_3) \tag{2}$$

with a fixed length $\boldsymbol{\varphi}^2 \equiv M^2 = m^2 + m_3^2$ and in the direction of the unit vector $\hat{\boldsymbol{\varphi}}(x)$. Now we can adiabatically create a vortex in $\hat{\varphi}_1 + i\hat{\varphi}_2$ by first rotating $\hat{\boldsymbol{\varphi}}$ from the initial uniform state "up" or "down" to $(0, 0, \mathrm{sgn}(m_3))$ and then flattening to the $m_3 = 0$ plane away from the center as illustrated in Fig. 1(a). This creates a *meron* (half a skyrmion) in $\hat{\boldsymbol{\varphi}}$ with the core pointing in the $(0, 0, \mathrm{sgn}(m_3))$ direction. On the lattice we can always take the limit of zero core size, thus recovering an ordinary U(1) vortex. In the continuum description under consideration here, however, there always remains a single point (coincident with the singularity in χ) where m_3 retains a nonzero value, $\pm M$. This value distinguishes between the two different ways of creating a vortex and will be seen below to have physical implications.

The utility of this procedure lies in the fact that the formation of a vortex is completely *smooth* and we can thus calculate the charge accumulated during this adiabatic process by a perturbative loop-expansion of the fermion current operator. On symmetry and dimensional grounds we may expect

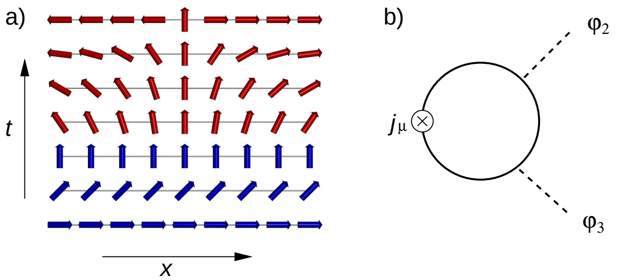

FIG. 1: (Color online) (a) Adiabatic formation of a vortex from a uniform, vortex-free configuration. The arrows represent the direction of $\hat{\boldsymbol{\varphi}}$ along the $y = 0$ cut through the vortex center. (b) Feynman diagram used to calculate the fermion current. Solid lines represent the fermion propagator $G(p) = (\not{p} - m)^{-1}$.

that the fermion current will take the form of the conserved topological current in the O(3) nonlinear σ model,

$$j^{\rm top}_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}\epsilon_{abc}\hat{\varphi}_a\partial_\nu\hat{\varphi}_b\partial_\lambda\hat{\varphi}_c, \tag{3}$$

up to an overall prefactor. As in Ref. 16 the correctness of this anticipation and the prefactor can be established by computing the expectation value of the fermion current $j_\mu = \bar{\psi}\gamma_\mu\psi$ in response to a small perturbation $\boldsymbol{\varphi}(x)$ imposed on top of a uniform background $\boldsymbol{\varphi}_0$, which we take to lie in the $(1, 0, 0)$ direction. A straightforward evaluation of the diagram in Fig. 1(b) yields $\langle j_\mu\rangle = (M/8\pi|M|^3)\epsilon_{\mu\nu\lambda}\partial_\nu\varphi_2\partial_\lambda\varphi_3$. Considering other permutations of external fields in the above diagram we identify $\langle j_\mu\rangle = j^{\rm top}_\mu$ for the general background field.

To calculate the charge associated with the meron configuration let us substitute $\boldsymbol{\varphi}(x)$ from Eq. (2) into (3) to obtain

$$\langle j_\mu\rangle = -\frac{1}{4\pi|M|}\epsilon_{\mu\nu\lambda}(\partial_\nu m_3)(\partial_\lambda\chi). \tag{4}$$

The total charge $Q = \int d^2\mathbf{x}\langle j_0\rangle = (4\pi|M|)^{-1}\int d^2\mathbf{x} m_3(\mathbf{x})\epsilon_{ij}\partial_i\partial_j\chi$ where we have used integration by parts and the fact that $m_3(\mathbf{x}) = 0$ far from the meron center. Recalling that for a static meron centered at the origin $\epsilon_{ij}\partial_i\partial_j\chi(\mathbf{x}) = 2\pi n\delta(\mathbf{x})$ with n the integer vorticity of χ we obtain $Q = \frac{1}{2}n\,\mathrm{sgn}(m_3(\mathbf{0}))$. This result has the satisfying property of not depending on the exact spatial profile of m_3 but only on its asymptotic value at the vortex center. This is in complete agreement with Refs. [3, 13, 14, 17] when we recall that the charge of a $n = \pm1$ vortex can be either $+\frac{1}{2}$ or $-\frac{1}{2}$ depending on whether or not is the zero mode occupied by an electron. In our present construction the sign of m_3 is seen to play the role of the electron occupancy, despite the fact that the zero mode never enters our discussion.

For future reference we also record the form of $\langle j_\mu\rangle$ in the limit of infinitely small meron core, i.e., $m_3(\mathbf{x}) = 0$ everywhere except $m_3(\mathbf{0}) = \pm M$. Eq. (4) becomes

$$\langle j_\mu\rangle = \frac{1}{4\pi}\mathrm{sgn}(m_3)\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi. \tag{5}$$

This generalizes naturally to the case of multiple vortices with $m_3 = \pm M$ at each vortex center.

Effective theory – We now turn to the discussion of statistics. Our strategy will be to find the low-energy effective theory for vortices in the Lagrangian (1). For the subsequent considerations it will be useful to modify the Lagrangian by including a coupling of the fermions to two gauge fields,

$$\mathcal{L} = \bar{\psi}\left(i\not{\partial} - \not{A} + \gamma_5\not{B} + me^{-i\chi\gamma_5}\right)\psi. \tag{6}$$

A_μ and B_μ couple minimally to the electric and the axial fermion currents, $j_\mu = \bar{\psi}\gamma_\mu\psi$ and $j_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi$, respectively, and can be therefore identified as the ordinary electromagnetic field and the chiral gauge field introduced in Ref. 17. In what follows we treat A and B as static external fields that help us keep track of the charge content of various fields.

We note that the coupling between vortices and fermions can be written as $m(\bar{\psi}_+e^{i\chi}\psi_- + \bar{\psi}_-e^{-i\chi}\psi_+)$ where $\psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi$ are the chiral components of the Dirac fermion. This suggests that in order to focus on the vortex degrees of freedom we perform a singular gauge transformation $\psi_\pm \to e^{\pm i\chi_\pm}\psi_\pm$, where we have defined an (as yet) arbitrary partitioning $\chi = \chi_+ + \chi_-$ of the vortex phase, such that upon encircling a vortex the spinor field remains single-valued [18]. (This simply means that the phase field associated with any given vortex is assigned to either χ_+ or χ_-.) Under this transformation the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - \not{a} + \gamma_5\not{b} + m)\,\psi, \tag{7}$$

with shifted gauge fields

$$a_\mu = A_\mu + \frac{1}{2}\partial_\mu(\chi_+ - \chi_-), \tag{8a}$$

$$b_\mu = B_\mu + \frac{1}{2}\partial_\mu(\chi_+ + \chi_-). \tag{8b}$$

A transformation of this type has been previously employed in the studies of the vortex state of a d-wave superconductor [18]. Its chief advantage is that it recasts a somewhat unwieldy Lagrangian (1) with a twisted mass term in the form of a gauge theory (7) which can be analyzed by standard field-theory methods. Specifically, our goal is to integrate out the fermi fields in (7) to arrive at the effective action in terms of the gauge fields which now encode the vortex degrees of freedom. As is usually the case this procedure encounters ultraviolet divergences. These must be regularized in a manner that is consistent with the symmetries of the underlying lattice model. In the following analysis we shall pay particular attention to the time reversal and parity which we expect to be preserved in the low-energy theory.

Symmetries – Besides Lorenz and gauge invariance, the low-energy theory also respects discrete symmetries. At $m = 0$ parity, $(t, x, y) \xrightarrow{\mathcal{P}} (t, -x, y)$, acts as $\psi \xrightarrow{\mathcal{P}} P\psi$ with $P \in \gamma_1\mathbf{u}(2)$ where $\mathbf{u}(2)$ is the group generated by $\{\mathbb{1}, i\gamma_3, \gamma_5, \gamma_3\gamma_5\}$. Charge conjugation, C, can be similarly worked out to be $\psi \xrightarrow{C} C\psi^*$ with $C \in \gamma_2\mathbf{u}(2)$. The antiunitary time-reversal operation, $(t, x, y) \xrightarrow{\mathcal{T}} (-t, x, y)$, is given by $\psi \xrightarrow{\mathcal{T}} T\psi$ with $T \in \gamma_1\mathbf{u}(2)$. The Lagrangian is odd under the unitary operation $(t, x, y) \xrightarrow{\mathcal{S}} (-t, x, y)$ that sends $\psi \xrightarrow{\mathcal{S}} S\psi$ with $S \in \gamma_0\mathbf{u}(2)$. That is, $\mathcal{S}$

anticommutes with the Hamiltonian. This is identified on the lattice as a sublattice symmetry [3] which renders the energy spectrum symmetric around zero.

The sense of the vortex is switched under parity: $\chi \stackrel{\mathcal{P}}{\to} -\chi$. This means that $(b_0, b_1, b_2) \stackrel{\mathcal{P}}{\to} (-b_0, b_1, -b_2)$. The gauge field a must, however, behave as a vector like the usual electromagnetic field, $(a_0, a_1, a_2) \stackrel{\mathcal{P}}{\to} (a_0, -a_1, a_2)$. So, we require that $\chi_\pm \stackrel{\mathcal{P}}{\to} -\chi_\mp$. Similarly we find $\chi_\pm \stackrel{\mathcal{C}}{\to} -\chi_\pm$ and $\chi_\pm \stackrel{\mathcal{T}}{\to} +\chi_\mp$. The latter implies $(b_0, b_1, b_2) \stackrel{\mathcal{T}}{\to} (-b_0, b_1, b_2)$ and $(a_0, a_1, a_2) \stackrel{\mathcal{T}}{\to} (a_0, -a_1, -a_2)$. These conditions also uniquely determine the operation of $\mathcal{P}, \mathcal{C}, \mathcal{T}$ and $\mathcal{S}$ on the fermi fields in the presence of vortices ($m \neq 0$) to be

$$P = \gamma_1\gamma_5; \;\; C = \gamma_2\gamma_5; \;\; T = \gamma_1\gamma_5; \;\; S = \gamma_0\gamma_3. \tag{9}$$

It follows that the bilinear $\bar\psi\gamma_3\psi$ is even under any operation, while $\bar\psi\gamma_3\gamma_5\psi$ is even under $\mathcal{S}$ and odd under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$. (In fact, the latter is true in the absence of vortices as well.)

Topological terms – Armed with the above analysis we can now ask what topological terms are allowed by symmetry in the low-energy effective action for a and b. It is easy to see that conventional CS terms $a\cdot(\partial\times a)$ and $b\cdot(\partial\times b)$ break both $\mathcal{P}$ and $\mathcal{T}$ and are thus prohibited. One can, however, construct a *mixed* CS term,

$$\mathcal{L}_{\rm CS} = \frac{\kappa}{4\pi} a\cdot(\partial\times b) \tag{10}$$

which obeys $\mathcal{P}$ and $\mathcal{T}$ and is thus allowed. The value of the coefficient κ is tied to the value of fractional charge. The simplest way to see this is to note that in view of Eq. (8a) varying the action $\mathcal{S}_{\rm CS} = \int {\rm d}t{\rm d}^2\mathbf{x}\mathcal{L}_{\rm CS}$ with respect to A_μ gives the fermion current,

$$\langle j_\mu\rangle = \left.\frac{\delta\mathcal{S}_{\rm CS}}{\delta A_\mu}\right|_{A,B=0} = \frac{\kappa}{8\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi. \tag{11}$$

Consistency with Eq. (5) then requires $\kappa = 2{\rm sgn}(m_3)$.

The mixed CS term (10) can also be obtained more directly from the Lagrangian (7) if we regularize it by adding a Pauli-Villars mass term. Based on our discussion of symmetries we choose to add the γ_3-mass term, which preserves both $\mathcal{T}$ and $\mathcal{P}$ but breaks $\mathcal{S}$. The physical amplitudes are found in a standard perturbative expansion using the Pauli-Villars subtraction, $\mathcal{A}_{\rm phys} = \mathcal{A}_{\rm reg}(m_3 = 0) - \mathcal{A}_{\rm reg}(m_3 \to \infty)$. Integrating out the Dirac fermions to one-loop order is a lengthy but ultimately straightforward exercise that yields [19]

$$\mathcal{L}_{\rm eff} = -\frac{\pi}{12|m|}(\partial\times a)^2 + \frac{|m|}{2\pi}b^2 + \frac{{\rm sgn}(m_3)}{2\pi}a\cdot(\partial\times b).$$

The last term is the mixed CS term obtained before. The $(\partial\times a)^2$ term describes the expected dielectric response of the Dirac medium. In the absence of the chiral gauge field the b^2 term becomes simply $(\partial\chi)^2$. This reflects the cost of spatial and temporal variations in $\chi(x)$ and encodes the usual logarithmic interaction between vortices. If the chiral gauge field is present and described at the bare level by a Maxwell term, then the interaction between vortices is exponentially screened at long distances, as in ordinary type-II superconductors. Note that m appears as the charge of a and at the same time as the mass of b. Therefore, by tuning $m \to 0$ we could expect to find a phase where a is massive and b is soft, hence the role of axial and regular currents is reversed. This would be a superconducting phase.

Exchange statistics – We now analyze the implications of the mixed CS term (10) for the vortex statistics. To this end we substitute a and b from Eqs. (8) into $\mathcal{L}_{\rm CS}$ and set $A = B = 0$,

$$\mathcal{L}_{\rm CS} = \frac{\kappa}{16\pi}(u_+\cdot\partial\times u_+ - u_-\cdot\partial\times u_-), \tag{12}$$

where we defined $u_{\pm\mu} = \partial_\mu\chi_\pm$. It is now easy to understand the exchange statistics. Take two vortices at $\mathbf{x}_1$ and $\mathbf{x}_2$ and let the second one go on a path C_2 around the first one, which remains static. Let us assume they both belong to the same $\pm$-partition and ${\rm sgn}(m_3) > 0$. In Eq. (12) we may then write $u_\pm = u_1 + u_2$ and $u_\mp = 0$ where $\frac{1}{2\pi}(\partial\times u_k)_\mu = ({\rm d}x_\mu/{\rm d}t)\delta^{(2)}(\mathbf{x}-\mathbf{x}_k(t))$ is the current density of the k-th vortex. The topological phase $e^{2i\theta}$ that is accumulated in this process can be calculated from the $u_1\cdot\partial\times u_2$ cross term in Eq. (12):

$$\begin{aligned} 2\theta &= \pm\frac{1}{2}\int {\rm d}^3x\delta^{(2)}(\mathbf{x}-\mathbf{x}_2(t))\frac{{\rm d}x_\mu}{{\rm d}t}u_{1\mu} \\ &= \pm\frac{1}{2}\oint_{C_2}{\rm d}\mathbf{x}_2\cdot\mathbf{u}_1 = \pm\pi. \end{aligned} \tag{13}$$

This means that two such vortices behave as *semions* with the statistical angle $\theta = \pm\pi/2$. Alternatively, if we have two vortices in two different $\pm$-partitions there is no cross term and they will be mutual bosons.

How can the assignment of vortices into the seemingly arbitrary $\pm$-partitions produce observable effects? The answer lies in the realization that this assignment actually entails a genuine physical distinction between vortices in the two partitions. This can be seen by computing the axial current by varying $\mathcal{S}_{\rm CS}$ with respect to B_μ, as in Eq. (11), to obtain $\langle j_{5\mu}\rangle = \frac{\kappa}{8\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda(\chi_+ - \chi_-)$. This implies that elementary vortices assigned to different partitions carry opposite axial charge $Q_5 = \int d^2\mathbf{x}\langle j_{50}\rangle = \pm\kappa/4$. If we had a chiral gauge field probe at our disposal, we could in principle detect the axial charge associated with a vortex just as we can detect its electric charge. In the absence of such a probe the axial charge still can affect the physics, e.g., by influencing the exchange statistics.

The Lagrangian (1) posesses an emergent global symmetry under the transformation $\psi \to e^{i\chi_0\gamma_5}\psi$ and $\chi \to \chi - 2\chi_0$. This guarantees conservation of the axial charge in all low-energy processes. When a vortex is created, e.g., in a pair-creation process, it is endowed by a particular value of the electric charge ($Q = \pm\frac{1}{2}$) and the axial charge ($Q_5 = \pm\frac{1}{2}$). These values then uniquely characterize the vortex; in particular they determine its exchange statistics. In order to better appreciate this key point it is useful to recall that $\psi_\pm$ represent the field operators associated with the two different Dirac nodes [3, 4]. Physical meaning of the axial charge then becomes clear from the expressions $j_0 = \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-$ and $j_{50} = \psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-$.

Since the nodes interchange under $\mathcal{T}$ and $\mathcal{P}$ the axial charge is *odd* under these operations. The spatial components of j_5 are closely related to the 'valley currents' recently studied in the context of graphene [20].

The above result (13) could also be understood using the following physical picture. Starting with two vortices in the +-partition we could choose to transfer one, say vortex 2, to the −-partition. The symmetric field, b, does not change but the anti-symmetric field, a, shifts by $-u_2$. This shift could be absorbed in A, which will then attach a 2π flux to vortex 2. As a result, we pick up an extra Aharonov-Bohm phase π by taking the flux 2π of vortex 2 around the charge $\frac{1}{2}$ of vortex 1. The statistics remains unchanged.

Doubled CS theory – It is possible to rewrite the Lagrangian (12) in a more familiar form by introducing two auxiliary gauge fields $\mathcal{A}_\pm$ mediating the statistical interaction between vortices,

$$\mathcal{L}_{\rm CS} = \sum_{\sigma=\pm}\left(-\frac{1}{\pi\kappa}\sigma\mathcal{A}_\sigma\cdot\partial\times\mathcal{A}_\sigma + j_\sigma\cdot\mathcal{A}_\sigma\right) \qquad (14)$$

where $j_\pm = \frac{1}{2\pi}(\partial\times\partial\chi_\pm)$ are currents associated with vortices in two partitions. Gaussian integration over $\mathcal{A}_\pm$ leads back to Eq. (12). We recognize the above Lagrangian as the level-2 doubled CS theory of Ref. 10, constructed there based on very general considerations.

Concluding remarks – The statistical angle $\pi/2$ of a vacant vortex means that fusing two vacant vortices must result in a boson (with statistical angle 2π). At first sight this seems counterintuitive, since the resulting charge is $\frac{1}{2}+\frac{1}{2} = +1$ which could be expected to be a regular hole, and thus a fermion. But there is a subtlety here. Fusing two vortices gives a double vortex which supports *two exact zero modes* [3, 4, 14]. Now, suppose we remove the hole by adding an electron, so we have a neutral vortex. Since the electron can go to either of the two modes there is a pseudospin-$\frac{1}{2}$ degree of freedom attached to the neutral vortex. By the spin-statistics theorem this indeed should be a *fermion*, consistent with the notion that the original double vortex is a charge-1 boson.

It is also interesting to contemplate exactly how our model evades the argument of Ref. 6. The crucial point is that since the axial charge is odd under time reversal the many-body ground state Ψ of the system with two vortices must be at least two-fold degenerate, with $\Psi^* = \mathcal{T}\Psi$ and Ψ mutually orthogonal. However, the absence of anyons in a $\mathcal{T}$-invariant system only follows from Ref. 6 when the state is *non-degenerate*. It is straightforward to generalize the argument to the doubly degenerate situation [19]. One finds that because of the additional structure introduced by the degeneracy, $\theta = \pm\pi/2$ semions are allowed in addition to fermions and bosons, in agreement with the results of the effective theory.

While this work was in the final stages a preprint by Chamon *et al.* [21] appeared, in which the O(3) topological current is derived consistent with our Eq. (3). Their conclusions regarding the exchange statistics, however, appear to disagree with ours.

Acknowledgment – The authors have benefited from discussions with C. Chamon, M.P.A. Fisher, R. Jackiw, E.-A. Kim, S.A. Kivelson, M. Lawler, S.H. Simon, G. Semenoff and K. Shtengel. This research has been supported by NSERC, CIAR NSF Grant No. PHY05-51164 (KITP) and the Killam Foundation. M.F. acnowledges hospitality of the Aspen Center for Physics.

[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[3] C.-Y. Hou, C. Chamon, and C. Mudry, Phys. Rev. Lett. **98**, 186809 (2007).
[4] B. Seradjeh, C. Weeks and M. Franz, arXiv:0706.1559.
[5] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
[6] S. A. Kivelson and D. S. Rokhsar Phys. Rev. Lett. **61**, 2630 (1988).
[7] A. Yu. Kitaev, Ann. Phys. **303**, 2 (2003).
[8] A.K. Geim and K.S. Novoselov, Nature Mat. **6**, 183 (2007).
[9] C. Weeks *et al.*, Nature Phys. (in press, cond-mat/0703001).
[10] M. Freedman *et al.*, Ann. Phys. **310** 428 (2004).
[11] G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
[12] D. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[13] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[14] R. Jackiw and P. Rossi, Nucl. Phys. B **190**, 681 (1981).
[15] E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).
[16] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[17] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. **98** 266402 (2007).
[18] M. Franz and Z. Tešanović, Phys. Rev. Lett. **84**, 554 (2000); *ibid.* **87**, 257003 (2001).
[19] B. Seradjeh and M. Franz, (unpublished).
[20] A. Rycerz, J. Tworzydlo and C.W.J. Beenakker, Nature Phys. **3**, 172 (2007).
[21] C. Chamon *et al.* arXiv:0707.0293.